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Reliability properties and applications of proportional reversed hazards in reversed relevation transform

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Abstract

The concept of reversed relevation transform was introduced by Di Crescenzo and Toomaj (2015). In this article, we study important reliability properties of the reversed relevation transform under the proportional reversed hazards assumption. The results of research on information measures are presented. Various ageing concepts and stochastic orders are discussed. A new flexible generalisation of the Fréchet distribution is introduced using the proposed transformation, and reliability properties and applications are discussed.

Key words: reversed relevation transform, proportional reversed hazards model, information measures, ageing properties, stochastic orders, quantile function.

1. Introduction

Let X denote lifetime of a component with cumulative distribution function (CDF) $F_X(\cdot)$. Suppose we randomly inspect the status of the component and let Y denote the random inspection time with CDF $F_Y(\cdot)$. Then the distribution function of the random variable $X[Y]$, which denotes the total time of X given that it is less than the random inspection time Y $(i.e.\ X|X \leq Y)$ is given by

$$F_{X[Y]}(x) = F_Y(x) + F_X(x) \int_x^\infty \frac{1}{F_X(t)}\, dF_Y(t), \quad x \geq 0. \tag{1.1}$$

Di Crescenzo and Toomaj (2015) called (1.1) the reversed relevation transform of X and Y. This can be viewed as a dual concept of the well-known relevation transform introduced and studied by Krakowski (1973). When X and Y are identically distributed $(i.e.\ F_Y(x) = F_X(x))$, then (1.1) becomes

$$F_{X[Y]}(x) = F_X(x)(1 - \log F_X(x)). \tag{1.2}$$

Di Crescenzo and Toomaj (2015) studied various properties of a sequence of random variables formed by the repeated application of the reversed relevation transform. Kayal (2016) introduced a generalization of the cumulative entropy (Di Crescenzo and Longobardi

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(2009)) using the idea of the reversed relevation transform. Some results connecting n-fold reversed relevation transform and generalized cumulative residual entropy (GCRE) (Psarrakos and Navarro (2013)) were given by Di Crescenzo and Toomaj (2017). A past inaccuracy measure based on the reversed relevation transform was studied by Di Crescenzo et al. (2018).

Let $f_X(x)$ denote the density function and $\bar{F}_X(x) = 1 - F_X(x)$ represent the survival function of a random variable X. Then the hazard rate of X, defined as $h_X(x) = \frac{f_X(x)}{\bar{F}_X(x)}$ gives the instantaneous rate of failure at any given time of the object under study. Another measure of peculiar interest is the reversed hazard rate, which is defined as $\lambda_X(x) = \frac{f_X(x)}{F_X(x)}$. In the context of lifetime studies, the reversed hazard rate has a crucial role when time elapsed since failure is a quantity of interest in order to predict the actual time of failure. Various properties and applications of the reversed hazard rate can be seen in Block et al. (1998), Chandra and Roy (2001), Gupta and Nanda (2001), Finkelstein (2002) and Chechile (2011). In a parallel system of independent and identically distributed components, we can see that the reversed hazard rate of the system lifetime is proportional to the reversed hazard rate of the lifetime of each component. Lehmann (1953) introduced the concept of the proportional reversed hazards model (PRHM) in contrast to the well-known proportional hazards model (PHM), which is commonly used in reliability theory and survival analysis. Let $\lambda_X(\cdot)$ and $\lambda_Y(\cdot)$ be the reversed hazard rates of X and Y respectively. Then Y is said to be the PRHM of X with proportionality constant θ if

$$\lambda_Y(x) = \theta \lambda_X(x), \quad \theta > 0. \tag{1.3}$$

An equivalent form of (1.3) is

$$F_Y(x) = (F_X(x))^{\theta}, \quad \theta > 0. \tag{1.4}$$

PRHM can accommodate non-monotonic hazard rates even though the baseline hazard rate is monotonic. Mudholkar and Srivastava (1993), Mudholkar et al. (1995), Mudholkar and Hutson (1996), Gupta et al. (1998), Gupta and Kundu (1999, 2001, 2002, 2007), Sarhan and Kundu (2009), Mahmoud and Alam (2010), Popović et al. (2022) and several other authors have studied the importance of PRHM model in various lifetime contexts. Moreover, certain characterization results, ageing properties and stochastic orders of the PRHM can be seen in Di Crescenzo (2000), Gupta and Wu (2001), Kundu and Gupta (2004), Gupta and Kundu (2007) and Shojaee and Babanezhad (2023). Under the assumption of PRHM between X and Y, we call the transform (1.1) as the proportional reversed hazards in the reversed relevation transform (PRHRRT). The aim of the present paper is to uncover special properties and applications of PRHRRT in reliability context. Throughout the paper, the terms increasing and decreasing are used in a wide sense, that is, a function g is increasing (decreasing) if $g(x) \leq (\geq)\, g(y)$ for all $0 < x < y$. Whenever we use a derivative, an expectation, or a conditional random variable, we are tacitly assuming that it exists.

The remainder of this article is organized as follows. In Section 2, the concept of PRHRRT model is introduced and its basic structural properties are studied. Various reliability properties and certain interesting results based on information measures are discussed in Section 3. Ageing properties and stochastic orders of PRHRRT are studied in Sections 4 and 5 respectively. In Section 6, we introduce a new generalization of the Fréchet distribution using the concept of PRHRRT and present its distributional properties and applications. Finally, Section 7 provides major conclusions of the study.

2. Proportional reversed hazards in reversed relevation transform

Let X and Y be two non-negative random variables with absolutely continuous CDFs $F_X(\cdot)$ and $F_Y(\cdot)$ respectively. Suppose Y is the PRHM of X, as defined in (1.4). Then the reversed relevation random variable $X[Y]$ has the distribution function of the form

$$F_{X[Y]}(x) = (F_X(x))^\theta + F_X(x) \int_x^\infty \frac{1}{F_X(t)} d\left(F_X(t)\right)^\theta, \quad x \geq 0,\ \theta > 0. \tag{2.1}$$

Di Crescenzo and Toomaj (2015) have showed that the reversed relevation transform is commutative under the assumption of PRHM ($i.e.$ $X[Y] \stackrel{d}{=} Y[X]$). When $\theta = 1$, (2.1) reduces to (1.2) and hence in the present study we assume that $\theta \neq 1$. We now establish an identity connecting the distribution functions of $X[Y]$ and the baseline random variable X.

Proposition 2.1. Let X and Y be two non-negative random variables with absolutely continuous CDFs $F_X(x)$ and $F_Y(x)$ respectively. Then Y is the PRHM of X if and only if $F_{X[Y]}(x)$ satisfies

$$F_{X[Y]}(x) = \frac{\theta F_X(x) - (F_X(x))^\theta}{\theta - 1}, \quad \theta > 0. \tag{2.2}$$

Proof. Let Y be the PRHM of X. Then from Di Crescenzo and Toomaj (2015) (Proposition 2), the identity (2.2) follows. Now, to prove the converse part, assume that

$$F_Y(x) + F_X(x) \int_x^\infty \frac{1}{F_X(t)} dF_Y(t) = \frac{\theta F_X(x) - (F_X(x))^\theta}{\theta - 1}.$$

Rearranging and taking the first derivative with respect to x on both sides gives

$$\begin{aligned} \frac{-f_Y(x)}{F_X(x)} &= \frac{1}{(\theta-1)^2 (F_X(x))^2} \Big[(\theta-1) F_X(x) \Big(\theta f_X(x) - \theta \left(F_X(x)\right)^{\theta-1} f_X(x) - \\ & (\theta-1) f_Y(x) \Big) - \Big(\theta F_X(x) - (F_X(x))^\theta - (\theta-1) F_Y(x) \Big) (\theta-1) f_X(x) \Big]. \end{aligned}$$

Upon simplification, we get

$$\frac{f_X(x) \left(F_X(x)\right)^\theta}{\left(F_X(x)\right)^2} = \frac{f_X(x) F_Y(x)}{\left(F_X(x)\right)^2} \implies F_Y(x) = \left(F_X(x)\right)^\theta, \quad \text{for all } x \geq 0,\ \theta > 0.$$

This completes the proof. □

Remark 2.1. The CDF of $X[Y]$ given in (2.2) can be represented in a mixture form as

$$F_{X[Y]}(x) = \phi F_X(x) + (1-\phi)\left(F_X(x)\right)^{\theta} = \phi F_X(x) + (1-\phi) F_Y(x), \tag{2.3}$$

where $\phi = \frac{\theta}{\theta-1}$ and one of the weights is negative depending on the value of ϕ.

Let $f_{X[Y]}(x)$ denote the density function of the random variable $X[Y]$. Then from (2.2), we get

$$f_{X[Y]}(x) = f_X(x)\left(\frac{\theta}{\theta-1}\left(1-\left(F_X(x)\right)^{\theta-1}\right)\right), \tag{2.4}$$

where $f_X(x)$ is the density function of X. An equivalent representation of (2.2) in terms of the survival function of X, Y and $X[Y]$ denoted respectively by $\bar{F}_X(\cdot)$, $\bar{F}_Y(\cdot)$ and $\bar{F}_{X[Y]}(\cdot)$ is as follows:

$$\bar{F}_{X[Y]}(x) = \frac{\theta \bar{F}_X(x) - \bar{F}_Y(x)}{\theta-1}. \tag{2.5}$$

Now, the expected value of $X[Y]$ can be evaluated as follows:

$$\begin{aligned} E(X[Y]) &= \int_0^{\infty} \bar{F}_{X[Y]}(x)\,dx = \frac{1}{\theta-1}\int_0^{\infty}\left(\theta \bar{F}_X(x) - \left(\bar{F}_X(x)\right)^{\theta}\right)dx \\ &= \frac{1}{\theta-1}\int_0^{\infty}\left(F_X(x) + \theta \bar{F}_X(x) - \theta \bar{F}_X(x) - \left(F_X(x)\right)^{\theta}\right)dx \\ &= E(X) + T_X(\theta), \end{aligned} \tag{2.6}$$

where $T_X(\theta) = \frac{1}{\theta-1}\int_0^{\infty}\left(F_X(x) - \left(F_X(x)\right)^{\theta}\right)dx,\ \theta > 0,\ \theta \neq 1$ is the cumulative Tsallis past entropy (CTE), introduced and studied by Calì et al. (2017). From (2.6), the CTE of X can be evaluated as

$$T_X(\theta) = E(X) - E(X[Y]), \tag{2.7}$$

The identity (2.7) can be used for constructing simple non-parametric estimator for $T_X(\theta)$ by using the estimators of $E(X)$ and $E(X[Y])$.

In reliability theory, PHM models plays a vital role in the comparison of the lifetime of two components. The random variables X and Y satisfy PHM if

$$h_Y(x) = \theta h_X(x),\ \ \theta > 0, \tag{2.8}$$

where $h_Y(x) = \frac{f_Y(x)}{\bar{F}_Y(x)}$ and $h_X(x) = \frac{f_X(x)}{\bar{F}_X(x)}$ are the hazard rates of X and Y respectively. An equivalent representation of (2.8) is $\bar{G}(x) = (\bar{F}(x))^{\theta},\ \ \theta > 0$. For more details on PHM, one could refer to Kalbfleisch and Prentice (2002) and Lawless (2003). When Y is the PRHM of X with proportionality constant θ, the CDF of $X[Y]$ has the form (2.2). Now, in the next proposition, for $\theta = 2$, we provide an interesting characterization of PRHRRT.

Proposition 2.2. Let X and Y be two lifetime random variables. Then, Y is the PRHM of X with proportionality constant $\theta = 2$ if and only if $X[Y]$ is the PHM of X with the same proportionality constant.

Proof. Suppose $F_Y(x) = (F_X(x))^2$. Then from (2.2), we have

$$F_{X[Y]}(x) = 2F_X(x) - (F_X(x))^2 \iff \bar{F}_{X[Y]}(x) = 1 - 2F_X(x) + (F_X(x))^2 \iff \bar{F}_{X[Y]}(x) = (\bar{F}_X(x))^2.$$

Thus, $X[Y]$ is the PHM of X with proportionality constant 2, which completes the proof. □

Remark 2.2. Suppose that the family of distributions of X is invariant under PHM (*i.e.* X and the corresponding PHM random variable belongs to the same family of distributions) and Y is the PRHM of X with proportionality constant $\theta = 2$. Then X is invariant under PRHRRT. For example, under the aforementioned setup, X is exponential with mean λ if and only if $X[Y]$ is exponential with mean $\frac{\lambda}{2}$.

The concept of odds ratio is well known in epidemiological research, serving as a measure of the approximate relative risk of an event, like disease or death, with or without a specific factor. Now, if we define X as an individual's lifespan, extending the event to encompass 'failure occurring by time x' for all $x > 0$, the odds function $\phi_X(\cdot)$, of X can be represented as follows:

$$\phi_X(x) = \frac{P(X > x)}{P(X \leq x)} = \frac{\bar{F}_X(x)}{F_X(x)}.$$

Note that the odds function is a decreasing function of x. For more details on properties and applications of odds functions, one could refer to Bennett (1983), Zimmer et al. (1998), Navarro et al. (2008), Khorashadizadeh et al. (2013), and the references therein.

Proposition 2.3. Let $\phi_X(x)$, $\phi_Y(x)$ and $\phi_{X[Y]}(x)$ denote the odds functions of X, Y and $X[Y]$ respectively. Then Y is the PRHRRT of X if and only if

$$\phi_{X[Y]}(x) = \frac{\theta\, \phi_X(x) - \phi_Y(x)\, (F_X(x))^{\theta-1}}{\theta - (F_X(x))^{\theta-1}}. \tag{2.9}$$

Proof. Under the assumption of PRHM, we have

$$\begin{aligned} \phi_{X[Y]}(x) &= \frac{\bar{F}_{X[Y]}(x)}{F_{X[Y]}(x)} = \frac{\theta \bar{F}_X(x) - (1 - (F_X(x))^\theta)}{\theta F_X(x) - (F_X(x))^\theta} \\ \iff \phi_{X[Y]}(x) &= \frac{\theta\, \phi_X(x) - \phi_Y(x)\, (F_X(x))^{\theta-1}}{\theta - (F_X(x))^{\theta-1}}, \end{aligned}$$

where $\phi_Y(x) = \frac{1-(F_X(x))^\theta}{(F_X(x))^\theta} = \frac{\bar{F}_Y(x)}{F_Y(x)}$. □

3. Reliability properties

The hazard rate of the random variable $X[Y]$ under the assumption of PRHM has the form

$$h_{X[Y]}(x) = \frac{f_{X[Y]}(x)}{1 - F_{X[Y]}(x)} = h_X(x) \left(\frac{\theta(1 - F_X(x)) \left(1 - (F_X(x))^{\theta-1}\right)}{\theta(1 - F_X(x)) - \left(1 - (F_X(x))^\theta\right)} \right)$$

$$= \quad h_X(x)\left(\frac{1-\frac{F_Y(x)}{F_X(x)}}{1-\frac{\bar{F}_Y(x)}{\theta\bar{F}_X(x)}}\right), \tag{3.1}$$

where $h_X(x)$ is the hazard rate of X.

Let $m_{X[Y]}(x)$ denote the mean residual life of the random variable $X[Y]$, defined by

$$m_{X[Y]}(x) = \frac{1}{\bar{F}_{X[Y]}(x)}\int_x^{\infty}\bar{F}_{X[Y]}(t)\,dt, \quad x>0. \tag{3.2}$$

On integrating (2.5) over the interval (x,∞), we get

$$\int_x^{\infty}\bar{F}_{X[Y]}(t)\,dt = \frac{\theta}{\theta-1}\int_x^{\infty}\bar{F}_X(t)\,dt - \frac{1}{\theta-1}\int_x^{\infty}\bar{F}_Y(t)\,dt, \quad x>0. \tag{3.3}$$

This gives

$$\begin{aligned} m_{X[Y]}(x)\bar{F}_{X[Y]}(x) &= \frac{\theta}{\theta-1}m_X(x)\bar{F}_X(x) - \frac{1}{\theta-1}m_Y(x)\bar{F}_Y(x) \\ \Longrightarrow m_{X[Y]}(x) &= \frac{\theta\ m_X(x)\bar{F}_X(x) - m_Y(x)\bar{F}_Y(x)}{\theta\bar{F}_X(x)-\bar{F}_Y(x)}, \end{aligned} \tag{3.4}$$

where $m_X(x)$ and $m_Y(x)$ are the mean residual life functions of X and Y respectively.

The mean inactivity time of $X[Y]$ has the form

$$\mu_{X[Y]}(x) = \frac{1}{F_{X[Y]}(x)}\int_0^x F_{X[Y]}(t)\,dt = \frac{\theta\ \mu_X(x) - (F_X(x))^{\theta-1}\mu_Y(x)}{\theta-(F_X(x))^{\theta-1}}, \tag{3.5}$$

where $\mu_X(x)$ and $\mu_Y(x)$ are the mean inactivity times of X and Y respectively.

Glaser's function of a random variable X with density function $f_X(x)$ is defined as $\eta_X(x) = -\frac{f'_X(x)}{f_X(x)}$ (Glaser (1980)), where prime denotes the first derivative. It is used as an alternative for the hazard rate in lifetime studies. Under the PRHM assumption between X and Y, the Glaser's function of $X[Y]$ satisfies the identity

$$\eta_{X[Y]}(x) = \eta_X(x)\left(\frac{(F_X(x))^{\theta}\left((\theta-1)(F_X(x))^2\right)-(F_X(x))^2 f'_X(x)}{F_X(x)f_X(x)f'_X(x)\left(F_X(x)-(F_X(x))^{\theta}\right)}\right). \tag{3.6}$$

The reversed hazard rate of $X[Y]$ is given by

$$\begin{aligned} \lambda_{X[Y]}(x) = \frac{f_{X[Y]}(x)}{F_{X[Y]}(x)} &= \theta\lambda_X(x)\left(\frac{F_X(x)-F_Y(x)}{\theta F_X(x)-F_Y(x)}\right) \\ &= \lambda_Y(x)\left(\frac{F_X(x)-F_Y(x)}{\theta F_X(x)-F_Y(x)}\right). \end{aligned} \tag{3.7}$$

The identities (2.5), (3.1), (3.4), (3.5), (3.6) and (3.7) are useful for obtaining the aforementioned reliability measures of $X[Y]$ from those of the baseline random variables X and Y. Moreover, we can make use of these identities to establish various ageing and ordering properties of $X[Y]$ without knowing the distribution of $X[Y]$.

3.1. Distorted representation

A distortion function, $q(u)$, is a non-decreasing function from $[0,1]$ to $[0,1]$, such that $q(0)=0$ and $q(1)=1$. Suppose that X and Y are two random variables with survival functions $\bar{F}_X(x)$ and $\bar{F}_Y(x)$ respectively. Then Y is said to be the distorted random variable of X if $\bar{F}_Y(x)=q(\bar{F}_X(x))$, where $q(u)$ is a distortion function. Denneberg (1990) introduced the concept of distortion functions, and later it gained wide popularity in the areas of actuarial science, insurance, economics, and risk analysis. The importance of distorted random variables in reliability studies has been pointed out by various researchers, such as Wang (1996), Sordo and Suárez-Llorens (2011), Navarro et al. (2013, 2014, 2016), Sordo et al. (2015) and Navarro (2022).

Proposition 3.1. If Y is the PRHM of X, then $X[Y]$ is a distorted random variable of X with distortion function

$$q(u)=\frac{1}{\theta-1}(\theta u-(1-(1-u)^{\theta})). \tag{3.8}$$

Proof. Since Y is the PRHM of X, from (2.5), the survival function $\bar{F}_{X[Y]}(x)$ can be expressed as

$$\bar{F}_{X[Y]}(x)=q(\bar{F}_X(x)), \quad \text{where } q(u)=\frac{1}{\theta-1}(\theta u-(1-(1-u)^{\theta})),\ u\in[0,1].$$

We can easily verify that $q(u)$ given in (3.8) is a distortion function. Thus, $X[Y]$ is a distorted random variable of X with distortion function $q(u)$. □

Note that the distortion function given in (3.8) is a convex function. Expressing $X[Y]$ as a distorted random variable of X will be useful in studying the preservation of various ageing properties from X to $X[Y]$ and establishing stochastic order relations between X and $X[Y]$. We consider this in Sections 4 and 5.

4. Ageing properties

In this section, we discuss some of the ageing properties of $X[Y]$ in connection with the baseline random variable X. Let X be a lifetime random variable with CDF $F_X(x)$, density function $f_X(x)$, survival function $\bar{F}_X(x)$, hazard rate $h_X(x)$ and reversed hazard rate $\lambda_X(x)$. We consider the following ageing properties;

(i) X is said to have an increasing (decreasing) hazard rate (*i.e.* IHR (DHR)) if the hazard rate $h_X(x)$ is increasing (decreasing).

(ii) X is said to have an increasing (decreasing) hazard rate average (*i.e.* IHRA (DHRA)) if $\frac{1}{x}\int_0^x h_X(u)\,du$ is increasing (decreasing).

(iii) X is new better (worse) than used (*i.e.* NBU (NWU)) if $\bar{F}_X(x+t) \leq (\geq)\ \bar{F}_X(x)\bar{F}_X(t)$, for all $x, t > 0$.

(iv) X is new better (worse) than used in hazard rate (*i.e.* NBUHR (NWUHR)) if $h_X(0) \leq (\geq) h_X(x)$, for all $x > 0$.

(v) X is said to have an increasing (decreasing) reversed hazard rate (*i.e.* IRHR (DRHR)) if $\lambda_X(x)$ is increasing (decreasing).

(vi) X is said to have an increasing (decreasing) likelihood ratio (*i.e.* ILR (DLR)) if $\log f_X(x)$ is concave (convex).

For more details on ageing properties and their applications, one may refer to Barlow and Proschan (1975), Lai and Xie (2006), Navarro (2022) and Breneman et al. (2022). In the context of coherent systems having independent and identical components, Navarro et al. (2014) showed that the system lifetime S is a distorted random variable of the component lifetime X with distortion function, say $q(u)$. Since $X[Y]$ is a distorted random variable of X, in the next proposition we present conditions for the preservation of reliability classes under the formation of PRHRRT by adopting results from Navarro et al. (2014).

Proposition 4.1. Let X and Y be two lifetime random variables, with CDFs $F_X(x)$ and $F_Y(x)$ respectively. Let $X[Y]$ be the reversed relevation of X and Y. Assume that Y is the PRHM of X. Then we have the following;

(i) For $\theta \geq 2$, X is IHR $\implies$ $X[Y]$ is IHR.

(ii) For $0 < \theta \leq 2$, X is DHR $\implies$ $X[Y]$ is DHR.

(iii) For $\theta > 0$, X is DRHR $\implies$ $X[Y]$ is DRHR.

(iv) For $0 < \theta \leq 2$, X is DLR $\implies$ $X[Y]$ is DLR.

(v) For $\theta > 0$, X is NWU $\implies$ $X[Y]$ is NWU.

(vi) For $\theta > 0$, X is DHRA $\implies$ $X[Y]$ is DHRA.

Proof. Consider the PRHRRT model given in (2.2). From (3.8), we have the distortion function connecting X and $X[Y]$ as $q(u) = \frac{(1-u)^\theta + \theta u - 1}{\theta - 1}$, $u \in [0,1]$. By recalling the results from Navarro et al. (2014) in the context of coherent systems, we have, if $\tau(u) = \frac{uq'(u)}{q(u)}$ is decreasing (increasing) in $(0,1)$ then the IHR (DHR) property will be preserved with respect to the distortion function $q(u)$. Thus, for proving (i) and (ii), we have to examine the monotonicity of $\tau(u) = \frac{uq'(u)}{q(u)} = \frac{u\left(\theta - \theta\ (1-u)^{\theta-1}\right)}{(1-u)^\theta + \theta\ u - 1}$. For this, we have

$$\tau'(u) = \frac{\theta\left(u\left(\left((\theta-1)^2 u - 2\right)(1-u)^\theta - u + 2\right) - \left((1-u)^\theta - 1\right)^2\right)}{(u-1)^2\left((1-u)^\theta + \theta u - 1\right)^2}. \tag{4.1}$$

The denominator of (4.1) is always non-negative, and by analyzing the numerator, we observe that the right-hand side is strictly positive for $0<\theta<2$, strictly negative for $\theta>2$ and zero for $\theta=2$. This completes the proof for (i) and (ii).
Again, from Navarro et al. (2014), we have the result that, if $k(u)=\frac{uq'(1-u)}{1-q(1-u)}$ is decreasing in $(0,1)$ then the DRHR property will be preserved from X to $X[Y]$. We have $k(u)=\frac{uq'(1-u)}{1-q(1-u)}=\frac{\theta\ (u^{\theta}-u)}{u^{\theta}-\theta\ u}$. On differentiating $k(u)$ with respect to u, we get

$$k'(u)=-\frac{(\theta-1)^2\theta u^{\theta}}{(u^{\theta}-\theta u)^2}\leq 0,\quad \text{for all } \theta>0 \text{ and } u\in(0,1).$$

Thus, $k(u)$ is decreasing in u and hence the proof of (iii) follows.

Let $l(u)=\frac{uq''(u)}{q'(u)}=-\frac{(\theta-1)u(1-u)^{\theta-1}}{(1-u)^{\theta}+u-1}$. From (3.8) we have

$$l'(u)=-\frac{(\theta-1)(1-u)^{\theta-2}\left((1-u)^{\theta}+u(\theta-\theta u+u)-1\right)}{\left((1-u)^{\theta}+u-1\right)^2},$$

which is non-negative when $0<\theta\leq 2$ for all $u\ \in\ (0,1)$. Now, from Navarro et al. (2014) (Proposition 2.2), proof of (iv) follows.

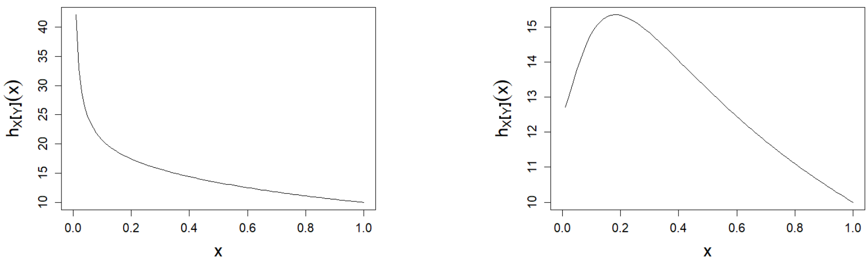
It is easy to verify that the distortion function $q(u)$ is super-multiplicative (*i.e.* $q(u\ v)\geq q(u)\ q(v)$, for all $0\leq u,\ v\leq 1$). This inequality with Proposition 2.7 of Navarro et al. (2014) completes the proof of (v).

Similarly $q(u)$ satisfies the inequality $q(u^a)\geq\ (q(u))^a$ for $0<a<1$. Now, proof of (vi) follows from Navarro et al. (2014) (Proposition 2.8). □

Example 4.1. Let X be a random variable having a Burr type-XII distribution, with CDF $F_X(x)=1-\left(\frac{1}{1+x}\right)^c,\ x>0,\ c>0$. Then the hazard rate of X is $h_X(x)=\frac{c}{1+x}$, which is decreasing for all parameter values. Suppose Y is the PRHM of X, then the hazard rate of $X[Y]$ has the form

$$h_{X[Y]}(x)=\frac{c\ \theta\left(\frac{1}{x+1}\right)^{c+1}\left(\left(1-\left(\frac{1}{x+1}\right)^c\right)^{\theta}+\left(\frac{1}{x+1}\right)^c-1\right)}{\left(\left(\frac{1}{x+1}\right)^c-1\right)\left(\theta\left(\frac{1}{x+1}\right)^c+\left(1-\left(\frac{1}{x+1}\right)^c\right)^{\theta}-1\right)}.$$

Figure 1(a) illustrate the preservation of DHR property when $0<\theta\leq 2$. For $\theta>2$, DHR property will not be preserved as shown in Figure 1(b). Observe that $X[Y]$ has DHR and Upside-down Bathtub (UBT) shaped hazard rates for various parameter combinations while the baseline is always DHR.



(a) $\theta = 0.5$ and $c = 10$ **(b)** $\theta = 5$ and $c = 10$

Figure 1: Plots of $h_{X[Y]}(x)$ for various parameter combinations.

5. Stochastic orders

Stochastic orders are used to compare the characteristics of two lifetime random variables. This section aims to provide different stochastic order relations between X and $X[Y]$. Let X and Y be two continuous lifetime random variables, with CDFs $F_X(x)$ and $F_Y(x)$ respectively. Let $f_X(x)$ and $f_Y(x)$ be the corresponding density functions. Then we have the following:

(i) X is smaller than Y in usual stochastic order, denoted by $X \leq_{st} Y$ if and only if $\bar{F}_X(x) \leq \bar{F}_Y(x)$ for all x.

(ii) X is smaller than Y in hazard rate order, denoted by $X \leq_{hr} Y$ if and only if $\frac{\bar{F}_Y(x)}{\bar{F}_X(x)}$ is increasing in x.

(iii) X is smaller than Y in likelihood ratio order, denoted by $X \leq_{lr} Y$ if and only if $\frac{f_Y(x)}{f_X(x)}$ is increasing in the set of union of their supports.

(iv) X is smaller than Y in increasing convex order, denoted by $X \leq_{icx} Y$ if and only if $\int_x^\infty \bar{F}_X(t)\, dt \leq \int_x^\infty \bar{F}_Y(t)\, dt$ for all x.

(v) X is smaller than Y in convex ordering, denoted by $X \leq_c Y$ if $F_Y^{-1}(F_X(x))$ is convex.

More properties and applications of stochastic orders can be seen in Shaked and Shanthikumar (2007), Belzunce et al. (2016) and Kochar (2022). Di Crescenzo and Toomaj (2015) showed that $X[Y] \leq_{st} X$. In the coming propositions, we establish interesting order properties between $X[Y]$, X and Y under the PRHM assumption between X and Y.

Proposition 5.1. Let Y be the PRHM of X and $X[Y]$ is the corresponding reversed relevation random variable. Then $X[Y] \leq_{lr} \min\{X, Y\}$

Proof. It is enough to show that $X[Y] \leq_{lr} X$ and $X[Y] \leq_{lr} Y$. For this, note that $X[Y]$ and X can be represented as distorted forms of X with respective distortion functions $q_1(u) = \frac{1}{\theta - 1}(\theta u - (1-(1-u)^\theta))$ and $q_2(u) = u$. By recalling the results from Navarro et al. (2013) in the context of stochastic orders between two coherent systems having identical components, we have

$$X[Y] \leq_{lr} (\geq_{lr}) X \text{ if and only if } \frac{q_1'(u)}{q_2'(u)} \text{ is increasing (decreasing) in } u \in (0,1). \quad (5.1)$$

Note that

$$\frac{d}{du}\left(\frac{q_1'(u)}{q_2'(u)}\right) = \frac{d}{du}\left(\frac{\theta - \theta(1-u)^{\theta-1}}{\theta - 1}\right) = \theta(1-u)^{\theta-2} > 0, \text{ for all } \theta > 0 \text{ and } u \in (0,1).$$

Thus, $\frac{q_1'(u)}{q_2'(u)}$ is increasing in $u \in (0,1)$ and thus from (5.1), we have $X[Y] \leq_{lr} X$.
In similar lines, we can form $X[Y]$ and Y by distorting Y using the distortion functions $r_1(u) = \frac{\theta\left(1-(1-u)^{\frac{1}{\theta}}\right)-u}{\theta-1}$ and $r_2(u) = u$ respectively. This gives

$$\frac{r_1'(u)}{r_2'(u)} = \frac{(1-u)^{\frac{1}{\theta}-1} - 1}{\theta - 1}.$$

Note that $\frac{d}{du}\left(\frac{r_1'(u)}{r_2'(u)}\right) = \frac{(1-u)^{\frac{1}{\theta}-2}}{\theta} > 0$, for all $\theta > 0$ and $u \in (0,1)$. The proof thus follows from (5.1). Now, since $X[Y] \leq_{lr} X$ and $X[Y] \leq_{lr} Y$, from Shaked and Shanthikumar (2007), we get $X[Y] \leq_{lr} \min\{X,Y\}$. This completes the proof. □

From Shaked and Shanthikumar (2007) and Proposition (5.1), we have the following implications.

$$X[Y] \leq_{lr} \min\{X,Y\} \implies X[Y] \leq_{hr} \min\{X,Y\} \implies X[Y] \leq_{st} \min\{X,Y\}.$$

Proposition 5.2. Let X_1 and X_2 be two lifetime random variables with distribution functions $F_1(x)$ and $F_2(x)$ respectively. Suppose Y_1 and Y_2 are the PRHM of X_1 and X_2 respectively with the same proportionality constant. Then the following properties hold:

(i) If $X_1 \leq_{st} X_2$, then $X_1[Y_1] \leq_{st} X_2[Y_2]$.

(ii) If $X_1 \leq_{hr} X_2$, then $X_1[Y_1] \leq_{hr} X_2[Y_2]$.

(iii) If $X_1 \leq_{icx} X_2$, then $X_1[Y_1] \leq_{icx} X_2[Y_2]$.

(iv) If $X_1 \leq_{lr} X_2$, then $X_1[Y_1] \leq_{lr} X_2[Y_2]$, for $\theta > 2$.

(v) If $X_1 \leq_{rhr} X_2$, then $X_1[Y_1] \leq_{rhr} X_2[Y_2]$.

Proof. The proof of (i) is intuitive from equation (2.2).
To prove (ii), we need to show that $\frac{u\, q'(u)}{q(u)}$ is decreasing in u. We have

$$\frac{d}{du}\left(\frac{uq'(u)}{q(u)}\right) = \frac{\theta\left(u\left(\left((\theta-1)^2u-2\right)(1-u)^{\theta}-u+2\right)-\left((1-u)^{\theta}-1\right)^2\right)}{(u-1)^2\left((1-u)^{\theta}+\theta u-1\right)^2} \leq 0,$$

for all $\theta > 0$, where $q(u)$ is the distortion function defined in (3.8). Then from Navarro et al. (2013) (Theorem 2.6), result (ii) follows.
Similarly (iii) follows from Navarro et al. (2013) (Theorem 2.6), since $q(u)$ is a convex function in $(0,1)$.

Again, from Navarro et al. (2013) we have the result that if $\frac{u\,q''(u)}{q'(u)}$ is non-negative and decreasing in u, then result (iv) holds. Now,

$$\frac{d}{du}\left(\frac{u\,q''(u)}{q'(u)}\right) = -\frac{(\theta-1)(1-u)^{\theta-2}\left((1-u)^{\theta}+u(\theta-\theta u+u)-1\right)}{\left((1-u)^{\theta}+u-1\right)^{2}} \leq 0,$$

for all $\theta \geq 2$. Then from Navarro et al. (2013) (Theorem 2.6), result (iv) follows.
Similarly, to prove (v), we use the result from Navarro et al. (2013) that, if $\frac{(1-u)\,q'(u)}{1-q(u)}$ is increasing in u, then result (v) holds. Note that

$$\frac{d}{du}\left(\frac{(1-u)\,q'(u)}{1-q(u)}\right) = \frac{\theta\left((1-u)^{\theta}+u-1\right)}{(1-u)^{\theta}+\theta(u-1)} > 0, \quad \text{for all } \theta > 0.$$

Then from Navarro et al. (2013) (Theorem 2.6), result (v) follows. □

Proposition 5.3. Let X and Y be two lifetime random variables with distribution functions $F_X(x)$ and $F_Y(x)$ respectively. If Y is the PRHM of X, then:

(i) $X[Y] \leq_c X$ for $\theta \geq 2$.

(ii) $X \leq_c X[Y]$ for $0 < \theta \leq 2$.

Proof. Sengupta and Deshpande (1994) showed that, for two non-negative random variables X and Y with hazard rates $h_X(x)$ and $h_Y(x)$ respectively, $X \leq_c Y$ if and only if $\frac{h_X(x)}{h_Y(x)}$ is non-decreasing in x, provided $h_Y(x) \neq 0$. To prove (i), we consider the function $s_1(x)$:

$$s_1(x) = \frac{h_{X[Y]}(x)}{h_X(x)} = \frac{\theta(F_X(x)-1)\left(F_X(x)-(F_X(x))^{\theta}\right)}{F_X(x)\left(\theta F_X(x)-(F_X(x))^{\theta}-\theta+1\right)}.$$

On differentiating with respect to x, we get

$$\frac{d}{dx}(s_1(x)) = \frac{\theta\left(-\left((\theta-1)^2(F_X(x))^2-2(\theta-2)\theta F_X(x)+(\theta-1)^2\right)(F_X(x))^{\theta}+F_X(x)^{2\theta}+(F_X(x))^2\right)f_X(x)}{(F_X(x))^2\left((F_X(x))^{\theta}-\theta F_X(x)+\theta-1\right)^2},$$

which is non-negative for $\theta \geq 2$. Thus, $X[Y] \leq_c X$ for $\theta \geq 2$.

Similarly, to prove (ii) we analyze the monotonicity of the function $s_2(x)$ defined by

$$s_2(x) = \frac{h_X(x)}{h_{X[Y]}(x)} = \frac{F(x)\left(-F(x)^{\theta}+\theta F(x)-\theta-1\right)}{\theta(F(x)-1)\left(F(x)-F(x)^{\theta}\right)}.$$

On differentiating with respect to x, we get

$$\frac{d}{dx}(s_2(x)) = \frac{\left((\theta-1)(F(x)-1)((\theta-1)F(x)-\theta-1)F(x)^{\theta}-F(x)^{2\theta}+F(x)^2\right)F'(x)}{\theta(F(x)-1)^2\left(F(x)-F(x)^{\theta}\right)^2},$$

which is non-negative for $0 < \theta \leq 2$. Thus, $X \leq_c X[Y]$ for $0 < \theta \leq 2$. □

6. Applications

In this section, we propose a generalization of the Fréchet distribution using the idea of PRHRRT. The Fréchet distribution is one of the well-known extreme value model. Extreme value theory is used to estimate the probability of extreme events and to develop strategies to reduce their effects. The classical theory of extremes deals with the distributional properties of the statistics $M_n = \max(X_1,\ldots,X_n)$ and $m_n = \min(X_1,\ldots,X_n)$ of i.i.d random variables $X_1,\ldots,X_n$. Gnedenko(1943) showed that the asymptotic distribution of M_n will be one of the three types of extreme value distributions. Type-I extreme value distribution is the Gumbel distribution, Type-II is the Fréchet or inverse Weibull distribution and Type-III is the reverse Weibull distribution. We have seen in Section 4 that PRHRRT can be used for constructing new lifetime models having more flexible hazard rates. We now assume the Fréchet distribution for the baseline random variable X and study various reliability properties of $X[Y]$. The two parameter Fréchet distribution has CDF

$$F_X(x) = \mathrm{e}^{-(\frac{\sigma}{x})^\alpha},\ x > 0,\ \sigma > 0,\ \alpha > 0.$$

Then the distribution function of the corresponding PRHRRT random variable $Z = X[Y]$ is obtained as

$$T_Z(x) = \frac{\theta\,\mathrm{e}^{-(\frac{\sigma}{x})^\alpha} - \left(\mathrm{e}^{-(\frac{\sigma}{x})^\alpha}\right)^\theta}{\theta - 1}, \quad x > 0,\ \sigma,\ \alpha > 0,\ \theta > 0. \tag{6.1}$$

We denote the model (6.1) as the PRHRR-F distribution. The r^{th} raw moment of PRHRR-F denoted by μ_r' is of the form

$$\mu_r' = \frac{\sigma^r\left(\theta - \theta^{\frac{r}{\alpha}}\right)\Gamma\left(1 - \frac{r}{\alpha}\right)}{\theta - 1}, \quad \alpha > r,\ r = 0,1,2,\ldots. \tag{6.2}$$

The moment generating function of Z is obtained as

$$M_Z(t) = \sum_{r=0}^{\infty} \mu_r' \frac{t^r}{r!}, \quad \alpha > r.$$

hazard rate of Z has the form

$$h_Z(x) = -\frac{\alpha\,\theta\left(\frac{\sigma}{x}\right)^\alpha\left(\left(\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^\alpha}\right)^{\theta-1} - 1\right)}{x\mathrm{e}^{\left(\frac{\sigma}{x}\right)^\alpha}\left(\theta + \left(\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^\alpha}\right)^\theta - 1\right) - \theta x}.$$

From Figure 2, we can observe that $h_Z(x)$ incorporates IHR, DHR and upside-down bathtub shapes for various parameter combinations.

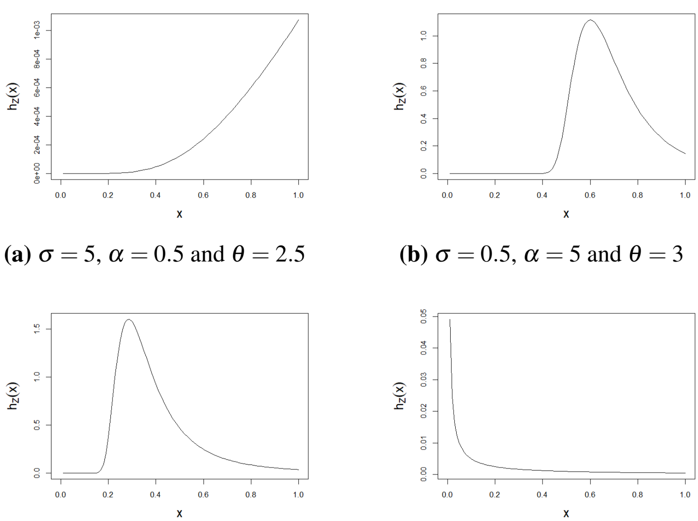
The estimation of unknown parameters of PRHRR-F (σ, α, θ) distribution has been carried out using the method of maximum likelihood. The log-likelihood function of the PRHRR-F for a given sample $x_1,\ldots,x_n$ of size n is

$$\log L(\sigma,\alpha,\theta|x_1,\ldots,x_n)=\sum_{i=1}^{n}\log\left(\frac{\alpha\,\theta\,\mathrm{e}^{-\left(\frac{\sigma}{x_i}\right)^{\alpha}}\left(\frac{\sigma}{x_i}\right)^{\alpha}\left(\left(\mathrm{e}^{-\left(\frac{\sigma}{x_i}\right)^{\alpha}}\right)^{\theta-1}-1\right)}{(1-\theta)x_i}\right).$$

The maximum likelihood estimators (MLE) ($\hat{\lambda}$, $\hat{\alpha}$ and $\hat{\theta}$) can be obtained by solving the equations $\frac{\partial \log L}{\partial \sigma}=0$, $\frac{\partial \log L}{\partial \alpha}=0$ and $\frac{\partial \log L}{\partial \theta}=0$ simultaneously. Since it is difficult to find a solution for this non-linear system of equations analytically, we have employed the Newton-Raphson iterative method to get a solution numerically. We have $\sqrt{n}(\hat{\Theta}-\Theta)$ follows multivariate normal distribution with zero mean and variance-covariance matrix $I^{-1}(\Theta)$, where $\Theta=(\sigma,\,\alpha,\,\theta)$ and $I(\Theta)$ denotes the Fisher information matrix. From this, the two-sided $100(1-\alpha)\%$ confidence interval for the parameters can be obtained as

$$\hat{\theta}_i \pm z_{\alpha/2}\sqrt{\frac{I_{ii}^{-1}(\Theta)}{n}}, \tag{6.3}$$

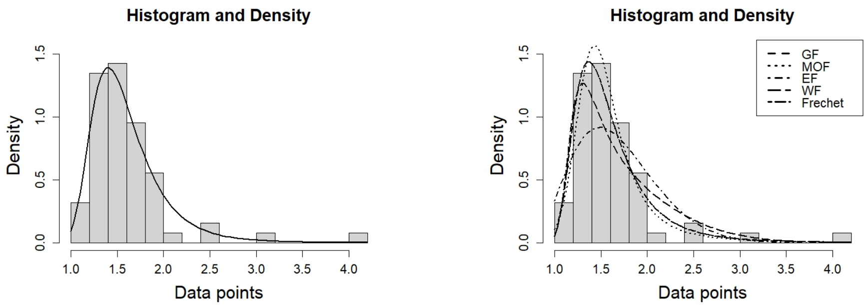
where $z_{\alpha/2}$ is the $\alpha/2^{th}$ percentile of the standard normal distribution and $I_{ii}^{-1}(\Theta)$ is the i^{th} diagonal element of $I^{-1}(\Theta)$, $i=1,\ldots,n$. When $I(\Theta)$ cannot be evaluated analytically, an efficient alternative is the observed Fisher information (OFI) introduced by Cox and Hinkley (1974).



(a) $\sigma=5$, $\alpha=0.5$ and $\theta=2.5$ **(b)** $\sigma=0.5$, $\alpha=5$ and $\theta=3$

(c) $\sigma=0.2$, $\alpha=3$ and $\theta=4$ **(d)** $\sigma=0.01$, $\alpha=0.01$ and $\theta=1.5$

Figure 2: Plots of $h_Z(x)$ for various parameter combinations.



(a) PRHRR-F distribution **(b)** Other competing models

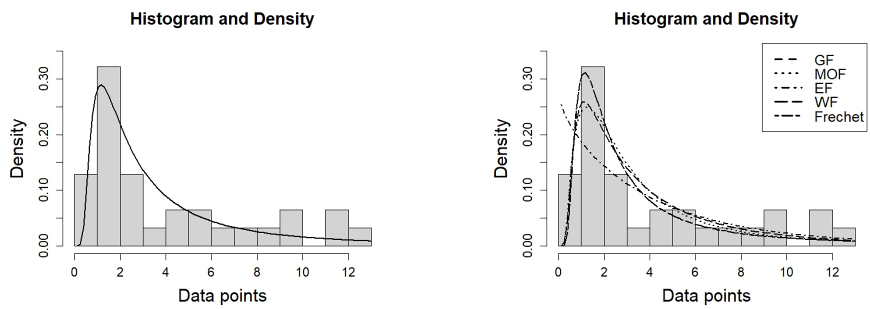
Figure 3: Histogram and Density plots for the first data set.

Table 1: Estimates, K-S statistics and *p*-values for the first data set.

Distributions	Estimates			K-S Statistics	*p*-value
PRHRR-F (σ, α, θ)	$\hat{\sigma} = 1.7527$	$\hat{\alpha} = 3.6595$	$\hat{\theta} = 0.9995$	0.0683	0.9114
GF (λ, α, β)	$\hat{\lambda} = 1.6737$	$\hat{\alpha} = 5.4376$	$\hat{\beta} = 0.3948$	0.0772	0.8185
MOF (α, β, λ)	$\hat{\alpha} = 1.4559$	$\hat{\beta} = 5.2227$	$\hat{\lambda} = 0.0023$	0.0813	0.7686
EF (α, β, λ)	$\hat{\alpha} = 1.7936$	$\hat{\beta} = 2.3223$	$\hat{\lambda} = 0.3016$	0.1739	0.0389
WF (α, β, λ)	$\hat{\alpha} = 1.6248$	$\hat{\beta} = 5.9372$	$\hat{\lambda} = 0.3750$	0.1659	0.0551
Fréchet (σ, α)		$\hat{\sigma} = 1.4108$	$\hat{\alpha} = 5.4377$	0.0772	0.8185

To show the applicability of the proposed model in situations other than reliability context, we next consider data that were reported in Hand et al. (1994). The data represents prices of 31 different children's wooden toys on sale in a Suffolk craft shop in April 1991. To show the efficiency of the proposed model over other competing alternatives, we carry out the K-S goodness of fit test. Maximum likelihood estimates and goodness of fit test results of the proposed model and other competing alternatives are listed in Table 2.

From Table 2 it is clear that, for the second data set, the PRHRR-F model outperforms other competing alternatives. The standard errors of $\hat{\sigma}$, $\hat{\alpha}$ and $\hat{\theta}$ are 0.0868, 0.0372 and 2.7654 respectively. The 95% confidence intervals for the model parameters σ, α and θ are (1.9347, 2.2751), (1.0177, 1.1636) and (6.6134, 17.4540) respectively. Figure 4 displays the observed histogram and fitted density functions. Q-Q plot is given in Figure 5(b). These two plots ensures the adequacy of the proposed model for the data.

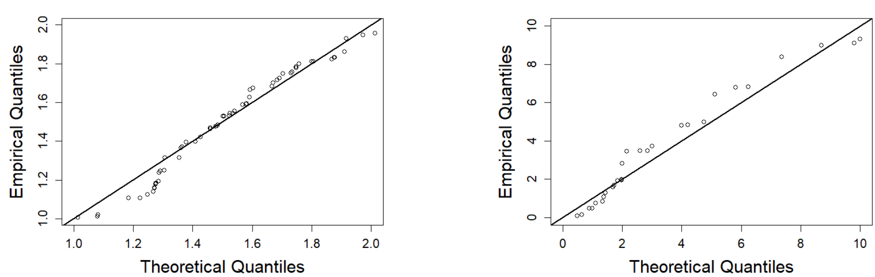


(a) PRHRR-F distribution **(b)** Other competing models

Figure 4: Histogram and Density plots for the second data set.

Table 2: Estimates, K-S statistics and p-values for the second data set.

Distributions	Estimates			K-S Statistics	p-value
PRHRR-F (σ,α,θ)	$\hat{\sigma}=2.1045$	$\hat{\alpha}=1.0906$	$\hat{\theta}=12.0434$	0.0821	0.9851
GF (λ,α,β)	$\hat{\lambda}=1.2321$	$\hat{\alpha}=1.2147$	$\hat{\beta}=1.6709$	0.0980	0.9271
MOF (α,β,λ)	$\hat{\alpha}=1.6728$	$\hat{\beta}=0.8776$	$\hat{\lambda}=6.4507$	0.1014	0.9074
EF (α,β,λ)	$\hat{\alpha}=2.6055\times10^{-14}$	$\hat{\beta}=0.9559$	$\hat{\lambda}=3.9062$	0.1392	0.5848
WF (α,β,λ)	$\hat{\alpha}=2.7451$	$\hat{\beta}=1.0389$	$\hat{\lambda}=0.7502$	0.1000	0.9156
Fréchet (σ,α)		$\hat{\sigma}=1.8802$	$\hat{\alpha}=1.2148$	0.0979	0.9271



(a) First data set **(b)** Second data set

Figure 5: Q-Q plots.

7. Conclusions

In this paper, we have presented the proportional reversed hazards in the reversed relevation transform as a special case of the reversed relevation transform. Its reliability properties and results based on entropy measures were discussed in detail. The ageing and stochastic ordering properties of the model were derived. Finally, we introduced the

PRHRR-F (σ, α, θ) model, studied its important characteristics and illustrated its practical applicability with the help of two real-life data sets.

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